Quantitative Methods

Mathematics	Marks
Basic mathematics	10
Financial mathematics	20
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Statistical methods	20
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Topics in this Chapter

1. **Simple Interest**
2. **Compound Interest**
3. **Annuities**
4. **Sinking fund**
5. **Time Value of money**
6. **Net Present Value in investment Appraisal (NPV)**
7. **Present value of Annuity**
8. **Present value of Perpetuity**
9. **Equivalent annual costs**
10. **Internal rate of return**

Interest

"Interest is the additional amount of money paid by the borrower to the lender for the use of money loaned to him".

When a person opens a savings account with a bank and deposits money into that account the bank will pay the person money for saving with them. Similarly, if a person borrows money from a bank the bank will expect that person to repay more than they borrowed. Money is not free to borrow. When a person or entity borrows money the lender
will charge interest.

The total interest associated with a loan is the difference between the total repayments and the amount borrowed.

There are two forms of interest:
- Simple interest; and
- Compound interest

1. Simple Interest

Simple interest is an interest which is charged on the fixed amount and it is not compounded.

Example 1

A person borrows Rs 10,000 at 10% with principle and interest to be repaid after 3 years.

Solution:

Amount owed at the start of year 1	10,000
Interest for year 1 (10%)	1,000
Interest for year 2 (10%)	1,000
Interest for year 3 (10%)	1,000
Total interest	3,000
Amount owed at the end of year 3	13,000

Formula: **A = P(1+rn)**

A=> Amount to be paid or received at the end of period n

P => Principal (amount borrowed or invested)
r => Period interest rate
n => Number of time periods that the loan is outstanding

Example 1

A = P(1+rn)

A= 10,000(1+0.1*3)

=13,000

Interest due on a loan **I = Prn**

Example 1

I = 10,000*.1*3

= 3,000

Practice questions

Q.1 A person borrows Rs.500,000 for 3 years at an interest rate of 8%. What must he pay to clear the loan at the end of this period?

Q.2 A person invests Rs. 50,000 for 5 years at an interest rate of 5%. What is the total interest received from this investment?

Answer:

1. **620,000**

2. **62,500**

r and n should be compared with like to like. Usually annual interest rate is given in question if interest rate

is annual and time is not annual then time should be adjusted accordingly.

For example, interest rate is 6% and time is 6 months.

Period Less than a year:

- If quarterly then time will be divided by 4.

- If Semi annually or bi annually then time will be divided by 2.

- If monthly then time will be divided by 12.

Example 2

A person borrows Rs 10,000 at 10% simple interest for 6 months?

Solution:

$A = P(1+rn)$

$A = 10,000(1+0.1*6/12)$

$=10,500$

Practice questions

Q.1 A person borrows Rs. 75,000 for 4 months at an interest rate of 8%.

Q.2 A person borrows Rs.60,000 at 8%. At the end of the loan he repays the loan in full with a cash transfer

of Rs.88,800. What was the duration of the loan?

Q.3 A person invests Rs.90,000 for 6 years. At the end of the loan she receives a cash transfer of Rs.122,400 in full and final settlement of the investment. What was the interest rate on the loan?

Answer:

1. **77,000**

2. **6 years**

3. **6%**

2. **Compound Interest**

Compound interest is where the annual interest is based on the amount borrowed plus interest accrued to date.

Example 3

Person borrows Rs 10,000 at 10% to be repaid after 3 years.

	Rs
Amount owed at the start of year 1	10,000
Interest for year 1 (10%)	<u>1,000</u>
Amount owed at the end of year 1 (start of year 2)	11,000
Interest for year 2 (10%)	<u>1,100</u>
Amount owed at the end of year 2 (start of year 3)	12,100
Interest for year 3 (10%)	<u>1,210</u>
Amount owed at the end of year 3	13,310

The closing balance of 13,310 must be repaid to the lender at the end of the third year.

Formula: $A = P(1+r)^n$

A=> Amount to be paid or received at the end of period n

P => Principal (amount borrowed or invested)
r => Period interest rate
n => Number of time periods that the loan is outstanding

Example 3

$A = P(1+r)^n$

$A = 10,000(1+0.1)^3$

$A = 13,310$

Compound interest for non annual periods

It is important to remember that the rate of interest *r* must be consistent with the length of the period *n*.
If interest is charged on an annual basis the annual interest rate must be used but if interest is charged on a six-monthly basis the six-monthly rate must be used.

Formula: $A = P(1+r/m)^{n*m}$

m => number of periods in a year.

Important points:

- If quarterly then 4 periods.

- If semi-annually or bi-annually then 2 periods.

- If monthly then 12 periods.

Example 4

A deposit 50,000 in a bank at an interest rate of 12% quarterly for 3 years. What will be the sum at the end of 3 years?

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Solution

A= 50,000(1+.12/4)^{3*4}

=71,285

Practice questions

1. A person borrows Rs.500,000 for years at an interest rate of 10%
2. A person invests Rs. 60,000 for 6 years at an interest rate of 6% compounded annually. What will the interest at the end of 6 years?
3. Ali borrows 10,000 for 4 years at an interest rate of 6% monthly what will he pays at end of 4 years?

Answers

1. 1,296,871
2. 25,111
3. 12,705

<u>**Nominal and effective rate:**</u>

Nominal rate is the rate which is shown on the face of loan however the actual rate can be changed. This is known as effective rate.

Formula:

R = (1+r)^{n} -1

R => Rate is for longer period

r => rate is for short period

n => No. of period in one long period

Example 5

A company wants to borrow Rs.1,000. It has been offered two different loans. Loan A charges interest at 10% per annum and loan B at 5% per 6 months. Which loan should it take?

Solution
- Loan A 10%
- Loan B

R = (1+r)^{n} -1

R = (1.05) ^2
R= 10.25%

<u>**Calculate n**</u>

Example 6

A man invested Rs.1,000 at 10% and received back Rs.2,595. How long was the money left on deposit?

Solution

$A = P \times (1 + r)^n$

$2{,}595 = 1{,}000 \times (1.1)^n$

$\dfrac{2{,}595}{1{,}000} = 1.1^n$

$2.595 = 1.1^n$

Take log of both sides log

$2.595 = n \log 1.1$

$\dfrac{\log 2.595}{\log 1.1} = n$

$n = 10$ years

Practice questions

1. A person borrows Rs.100,000 at an interest rate of 8% compounding annually. He repays Rs. 185,100 in full and final settlement at the end of the loan period. What is the duration of the loan?

2. A person borrows Rs. 100,000 at 5%. How long would it take the amount owed to double?
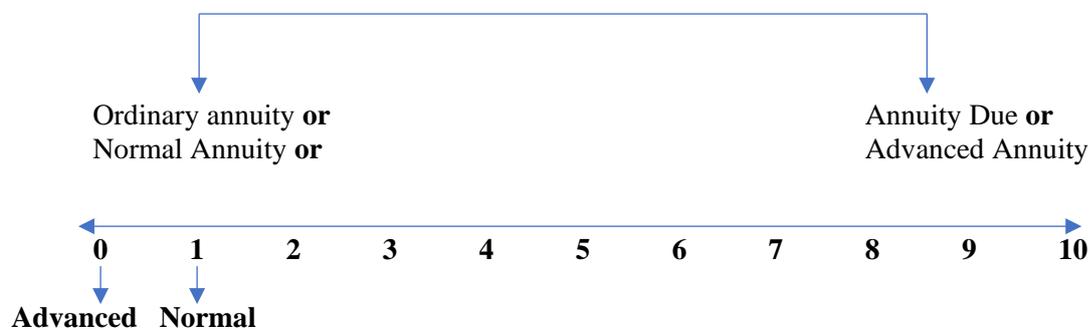
Answers

1. 8 years
2. 14.2 years

3. Annuities

Feature of annuities are:
- It is a series of payment
- At a regular interval
- For definite period

Example: 3,000 per month, 4,000 after every quarter etc.

Types of Annuity:

Ordinary annuity **or**   Annuity Due **or**
Normal Annuity **or**   Advanced Annuity

0   1   2   3   4   5   6   7   8   9   10

Advanced   Normal

Formula:

1. Ordinary Annuity   2.   Annuity Due

$$A = X \frac{(1+r)^n - 1}{r}$$ $$A = X \frac{(1+r)^n - 1}{r} \times (1+r)$$

A => sum of amount at the end of period.
X => Payment at regular interval
R => Interest rate
N => No. of period

Example 7

A savings scheme involves investing Rs.10,000 per annum for 5 years (on the last day of the year). If the interest rate is 10% what is the sum to be received at the end of the 5 years?

Solution:

$$A = X \frac{(1+r)^n - 1}{r}$$

$$= 10,000 \frac{(1+0.1)^5 - 1}{0.1}$$

$$= 61,051$$

Example 8

A savings scheme involves investing Rs.10,000 per annum for 5 years (on the first day of the year). If the interest rate is 10% what is the sum to be received at the end of the 5 years?

Solution:

$$A = X \frac{(1+r)^n - 1}{r} \times (1+r)$$

$$= 10,000 \frac{(1+0.1)^5 - 1}{0.1} \times (1+.1)$$

$$= 67,156$$

4. Sinking Fund:

A company wish to set a side a fixed amount of money at a regular interval to achieve a specific sum after some period. This is known as sinking fund.

Formula:

$$A = X \frac{(1+r)^n - 1}{r}$$

A => sum required at the end of period.
X => Payment at regular interval
R => Interest rate
N => No. of period

Example 9

A company will have to pay Rs.5,000,000 to replace a machine in 5 years. The company wishes to save up to fund the new machine by making a series of equal payments into an account which pays interest of 8%. The payments are to be made at the end of the year and then at each year end thereafter. What fixed annual amount must be set aside so that the company saves Rs.5,000,000?

Solution:

$$A = X \frac{(1+r)^n - 1}{r}$$

$$5,000,000 = X \frac{(1+0.08)^5 - 1}{0.08}$$

$$5,000,000 = X (5.867)$$

$$\frac{5,000,000}{(5.867)} = X$$

852,282

Practice questions

1. A business wishes to start a sinking fund to meet a future debt repayment of Rs. 100,000,000 dues in 10 years. What fixed amount must be invested every 6 months if the annual interest rate is 10% compounding semi-annually if the first payment is to be made in 6 months?

2. A man wants to save to meet the expense of his son going to university. He intends to puts Rs. 50,000 into a savings account at the end of each of the next 10 years. The account pays interest of 7%. What will be the balance on the account at the end of the 10-year period?

Answers

1. 3,024,803
2. 690,714

5. **Time value of money?**

One of the basic principles of finance is that a sum of money today is worth more than the same sum in the future. If offered a choice between receiving Rs10,000 today or in 1 year's time a person would choose today.
- Compounding is the method which calculates the **future value** of sum invested today.
- Discounting is a method which calculates the **present value** of future sum. Discounting is the reverse of compounding.

Discounting Formula:

$P = A (1+r)^{-n}$

Example 10

A person expects to receive Rs 13,310 in 3 years. If the person faces an interest rate of 10% what is the present value of this amount?

Solution

$A = P(1+r)^{-n}$

$P = 13,310 (1+0.1)^{-3}$

$P = 10,000$

It is important to realise that the present value of a cash flow is the equivalent of its future value. Using the above example to illustrate this, Rs 10,000 today is exactly the same as Rs 13,310 in 3 years at an interest rate of 10%. The person in the example would be indifferent between the two amounts. He would look on them as being identical.

Comparison of two or more investment or loan:

Example 11

A borrower is due to repay a loan of Rs 150,000 in 3 years. He has offered to pay an extra Rs 20,000 as long as he can repay after 5 years. The lender faces interest rates of 10%. Is the offer acceptable?

Solution

Option I

$A = P(1+r)^{-n}$

$P = 150,000 (1+0.1)^{-3}$

$P = 112,697$

Option II

$A = P(1+r)^{-n}$

$P = 170,000 (1+0.1)^{-5}$

$P = 105,557$

Decision:

Option I is better. It has more worth now.

Practice Questions

1. An investor wants to make a return on his investments of at least 7% per year. He has been offered the chance to invest in a bond that will cost Rs 100,000 and will pay Rs 150,000 at the end of five years. In order to earn Rs 150,000 after five years at an interest rate of 7% the amount of his investment now would need to be:

2. Bank A offers to earn on investment certificates Rs. 150,000 after 3 years if Ali invest 100,000 now and bank B offers to earn on investment certificates Rs. 170,000 after 5 years if Ali invest 100,000 now. What is the interest rate offering both the bank?

Answer

1. 106,948 should invest for earning 150,000 however he has to invest only 100,000 so he earns more than 7%.
2. Bank A 14.47% and Bank B 11.20%

Illustration: Discount tables (extract)

(Full tables are given as an appendix to this text).

Discount rates (r)

(n)	5%	6%	7%	8%	9%	10%
1	0.952	0.943	0.935	0.926	0.917	0.909
2	0.907	0.890	0.873	0.857	0.842	0.826
3	0.864	0.840	0.816	0.794	0.772	0.751
4	0.823	0.792	**0.763**	0.735	0.708	0.683

Where:

n = number of periods

How to discount table?

Example 12

Calculate the present value of Rs 60,000 received in 4 years assuming a cost of capital of 7%?

Solution

P = 60,000 * 0.763
P = 45,780

6. Net Present Value (NPV)

The cost of capital "r" is the return required by the investor or company. For example, we purchase machine for Rs. 100,000 now and we earn cashflow of Rs. 30,000 in 5 years. Sum of cashflow will be 150,000. If we compare 150,000 with 100,000, we can not say that we earn profit or loss because 100,000 is pay now and cashflow earn in five years times which has no same value today.
In this situation concept of time value of money.

Example 13

XYZ purchase machine today for Rs. 150,000. XYZ earns 30,000 in year 1 and after year 1 cashflow will be reduced by 2,000. After 5 years it has residual value (Scrap value) 20,000. Investor required rate of return is 10%. What will be NPV now?

Solution

Description	0	1	2	3	4	5
initial Investment	- 100,000					
Scrap value						20,000
Cash flow		30,000	28,000	26,000	24,000	22,000
	-100,000	30,000	28,000	26,000	24,000	42,000

Discount factor	1	0.909	0.826	0.751	0.683	0.621
Present value	- 100,000	27,270	23,128	19,526	16,392	26,082
NPV	12,398					

Practice Questions

1. A company is considering whether to invest in a new item of equipment costing Rs.53,000 to make a new product. The product would have a four-year life, and the estimated cash profits over the four-year period are as follows.

Year	Rs.
1	17,000
2	25,000
3	16,000
4	12,000

   Calculate the NPV of the project using a discount rate of 11%?
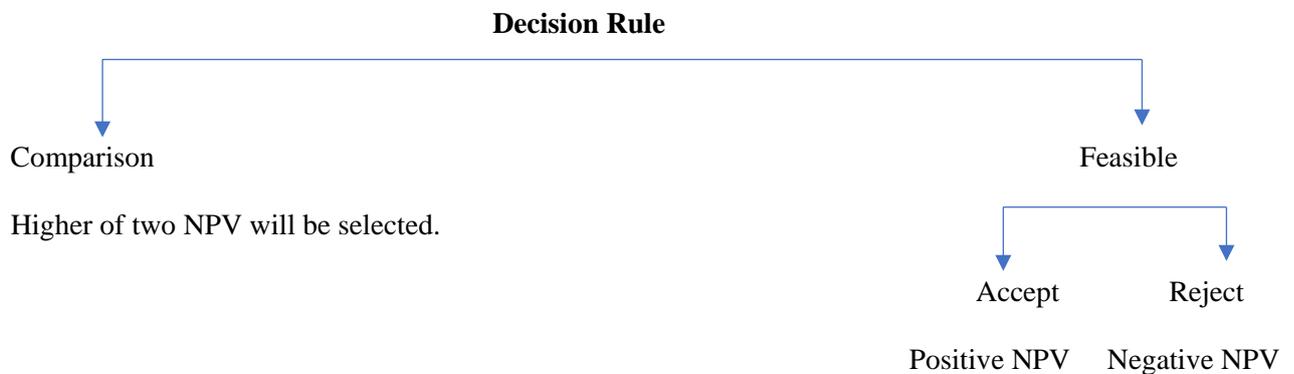
2. A company is considering whether to invest in a new item of equipment costing Rs.65,000 to make a new product. The product would have a three-year life, and the estimated cash profits over this period are as follows.

Year	Rs.
1	27,000
2	31,000
3	15,000

   Calculate the NPV of the project using a discount rate of 8%?

Answer
1. 2,210
2. (1,515)

Decision Rule

Comparison   Feasible

Higher of two NPV will be selected.

                                            Accept   Reject

                                    Positive NPV   Negative NPV

Rules of NPV

1. Initial investment should be taken at 0 unless otherwise told.
2. Cash flow arising during the year. It will be assumed at the end of year.
3. Cashflow arises beginning of the year. It will be assumed it arises at the end of previous year.

7. Discounting Annuity

Normal / ordinary

$$\frac{1 - (1 + r)^{-n}}{r}$$

Advanced / Annuity due

$$\frac{1 - (1 + r)^{-n}}{r} + 1$$

Example 14

XYZ will receive 10,000 each annually for 5 years at the rate of 10%. What will be present value if 10,000 receive
- At the end of year
- Beginning of year

Solution

Present Value = Annual CF * Annuity Factor

- = $10,000 \times 3.791$
  = 37,910

- = $10,000 \times (3.170 + 1)$
  = $10,000 \times 4.170$
  = 41,700

Discount rates (r)

(n)	5%	6%	7%	8%	9%	10%
1	0.952	0.943	0.935	0.926	0.917	0.909
2	1.859	1.833	1.808	1.783	1.759	1.736
3	2.723	2.673	2.624	2.577	**2.531**	2.487
4	3.546	3.465	3.387	3.312	3.240	3.170
5	4.329	4.212	4.100	3.993	3.890	3.791

Where:
n = number of periods

Note: Solve this question with formula.

Example 14

A company is considering whether to invest in a project which would involve the purchase of machinery with a life of five years. The machine would cost Rs.356,000 and would have a net disposal value of Rs.56,000 at the end of Year 5. The project would earn annual cash flows (receipts minus payments) of Rs.100,000. Calculate the NPV of the project using a discount rate of 10 %.

Solution

|	Period	Cash flow	Discount factor (10%)	Present value
Description				
initial Investment	0	- 356,000	1	- 356,000
Scrap value	5	56,000	0.621	34,776
Cash flow	1-5	100,000	3.791	379,100
NPV				57,876

Practice Questions

1. A company is considering whether to invest in a project which would involve the purchase of machinery with a life of four years. The machine would cost Rs.1,616,000 and would have a net disposal value of Rs.301,000 at the end of Year 4. The project would earn annual cash flows (receipts minus payments) of Rs.500,000. Calculate the NPV of the project using a discount rate of 10%

Answer
1. 174,519

8. Perpetuities

A perpetuity is a constant annual cash flow 'forever'.

Normal   Advanced

$\text{Annual Cash Flow} \times \dfrac{1}{r}$ $\text{Annual Cash Flow} \times \left(\dfrac{1}{r} + 1\right)$
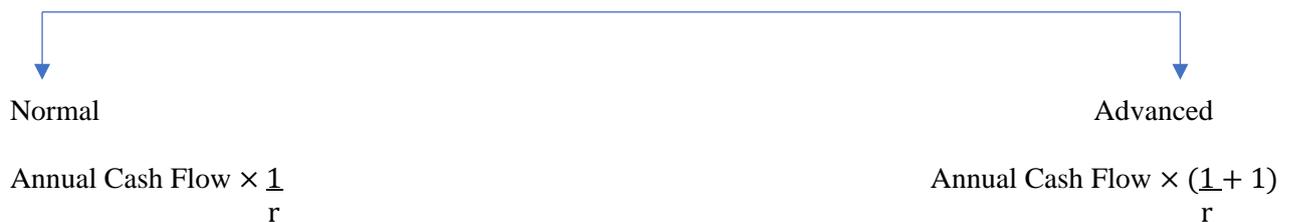
Example 14

2,000 in perpetuity, starting in Year 1 cost of capital = 8%
- At end of year
- Beginning of year

Solution

- = 2,000 × 1/.08
  = 25,000

- =2,000 × (1/.08 + 1)
  =27,000

9. Equivalent annual costs

An annuity is multiplied by an annuity factor to give the present value of the annuity. This can work in reverse. If the present value is known it can be divided by the annuity factor to give the annual cash flow for a given period that would give rise to it.

Example 15

What is the present value of 10,000 per annum from t1 to t5 at 10%?

Solution

Time	Cash flow	Discount factor	Present value
1 to 5	10,000	3.791	37,910

What annual cash flow from t1 to t5 at 10% would give a present value of 37,910?

Divide by the 5 years, 10% annuity factor

$$\frac{37,910}{3.791} = 10,000$$

Example 16

A company borrows Rs 10,000,000. This to be repaid by 5 equal annual payments at an interest rate of 8%. Calculate the payments?

Solution

Amount borrowed	10,000,000
Divide by the 5 years, 8% annuity factor	3.993
Annual repayment	2,504,383

Sinking Fund (Alternate Approach)

Step 1: Calculate the present value of the amount required in n years.
Step 2: Calculate the equivalent annual cash flows that result in this present value

Example 9 (Alternate Approach)

A company will have to pay Rs.5,000,000 to replace a machine in 5 years. The company wishes to save up to fund the new machine by making a series of equal payments into an account which pays interest of 8%. The payments are to be made at the end of the year and then at each year end thereafter. What fixed annual amount must be set aside so that the company saves Rs.5,000,000?
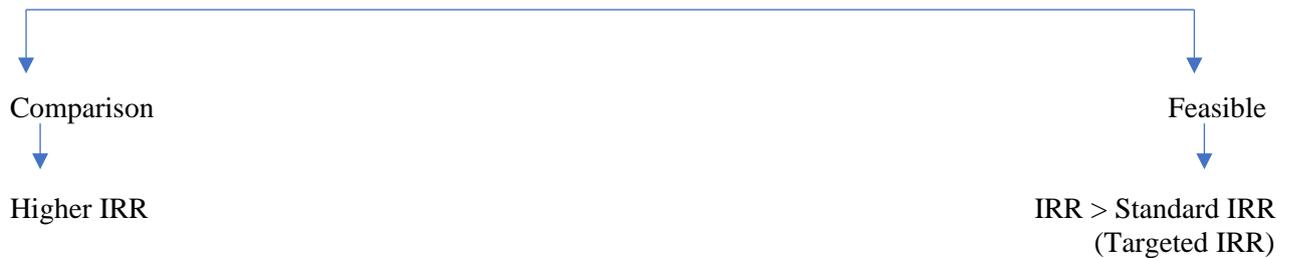
Solution

Step 1:
$= 5,000,000 \ (1.08)^{-5}$
$= 3,402,916$

Step 2:

Present value	3,402,916
Divide by the 5 years, 8% annuity factor	3.993
Annual repayment	852,220

10. *Internal rate of return (IRR)*

The internal rate of return is therefore the discount rate that will give a net present value = Rs.0.

Comparison   Feasible

Higher IRR   IRR > Standard IRR
                                          (Targeted IRR)

Formula

$$IRR = L\% + \left(\frac{NPV_L}{NPV_L - NPV_H}\right) \times (H - L)\%$$

Example 16

A business requires a minimum expected rate of return of 12% on its investments.

A proposed capital investment has the following expected cash flows.

Year	Cash flow	Discount factor at 10%	Present value at 10%	Discount factor at 15%	Present value at 15%
0	(80,000)	1.000	(80,000)	1,000	(80,000)
1	20,000	0.909	18,180	0.870	17,400
2	36,000	0.826	29,736	0.756	27,216
3	30,000	0.751	22,530	0.658	19,740
4	17,000	0.683	11,611	0.572	9,724
NPV			+ 2,057		(5,920)

Using

$$IRR = L\% + \left(\frac{NPV_L}{NPV_L - NPV_H}\right) \times (H - L)\%$$

$$IRR = 10\% + \left(\frac{2,057}{2,057 - (-5,920)}\right) \times (15 - 10)\%$$

$$IRR = 10\% + \left(\frac{2,057}{2,057 + 5,920}\right) \times 5\%$$

$$IRR = 10\% + \left(\frac{2,057}{7,977}\right) \times 5\%$$

$$IRR = 10\% + 0.258 \times 5\% = 10\% + 1.3\%$$

$$IRR = 11.3$$

Conclusion   The IRR of the project (11.3%) is less than the target return (12%).

The project should be rejected.

Practice Questions

1   The following information is about a project.

Year	Rs.
0	(53,000)
1	17,000
2	25,000
3	16,000
4	12,000

This project has an NPV of Rs.2,210 at a discount rate of 11%

Estimate the IRR of the project.

2   The following information is about a project.

Year	Rs.
0	(65,000)
1	27,000
2	31,000
3	15,000

This project has an NPV of Rs.(1,515) at a discount rate of 8%

Estimate the IRR of the project.

Answers

1. 12.7% (at 15%)
2. 6.6% (at 15%)